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TECHNICAL MEMORANDUM 1245

LIFT FORCE OF AN ARROW-SHAPED WING

By M. I. Gurevich

Translation

"O Pod'emnoi Sile Strelovidnogo Kryla v Sverkhzvukovom Potoke."
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LIFT FORCE OF AN ARROW-SHAPED WING*

By M. I. Gurevich

1. STATEMENT OF THE PROBLEM AND INTRODUCTION

The flow about a conical body of an ideal compressible fluid is considered. Assume that the velocity of the oncoming flow at infinity W is directed along the z-axis. The system of Cartesian coordinates x, y, z with origin at the vertex of the cone O is shown in figure 1. From the considerations of the dimensional theory, it may be found that along any ray issuing from O the components of the velocity u, v, W+w along the coordinate axes will maintain a constant value. It is further assumed that the conical body has such shape and disposition relative to the flow that u, v, and w are small in comparison with W. The equation of continuity can then, as is known, be given in the form

$$F_{xx} + F_{yy} - F_{zz}(M^2 - 1) = 0 \qquad (1.1)$$

where F may be considered either as the velocity potential φ or the components of the velocity u, v, w; $M = W/a$ is the Mach number, where a is the velocity of sound at infinity upstream of the body. It is assumed that the flow is everywhere supersonic $(M > 1)$.

In view of the linearity of equation (1.1), conical flows with different vertices may be superposed on one another. Equation (1.1) is of the hyperbolic type, which means that each point of the flow may have an effect only on the points located within its Mach cone; by a known method, the body may be transformed without affecting the flow about its remaining forward part. It must be borne in mind, however, that the Mach cones constructed for those points of the body at which the cut is made must nowhere intersect the remaining parts of the body. A. Busemann (reference 1) pointed out the analogy that exists between the problem of conical flows at supersonic velocity and small additional velocities and the two-dimensional problem for subsonic flow in the statement of the problem by S. A. Chaplygin (reference 2), which becomes exact when the equation of state of the gas has the form

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*"O Pod'emnoi Sile Strelovidnogo Kryla v Sverkhzvukovom Potoke." Prikladnaya Matematika i Mekhanika, vol. X, 1946, pp. 513-520.

$$p = \frac{C_1}{\rho} + C_2$$

where p is the pressure, ρ the density of the fluid, and C_1 and C_2 are constants. Making use of the considerations of Chaplygin, Busemann reduced the determination of u, v, w within the Mach cone to the solution of a Laplace equation in the two variables. In sections 2, 3, and 4, a direct derivation is given of the formulas of Busemann needed herein and the remaining sections are devoted to the investigation of the lift of an arrow-shaped wing.

2. TRANSFORMATION OF EQUATIONS OF CONTINUITY

For the conical flow, the components u, v, w can be represented as functions of the coordinate ratios $\xi = x/z$, $\eta = y/z$. The magnitudes ξ and η may be considered as the Cartesian coordinates in a plane perpendicular to the z-axis and intersecting it at the distance 1 from the origin (fig. 1). The Mach cone for the point 0 cuts out in the ξ,η plane a circle of radius $A = 1/\sqrt{M^2 - 1}$.

It is assumed for definiteness that $F = w$. It is readily shown that equation (1.1) in the variables ξ,η will, within the circle of radius A, be of the elliptic type and outside it of the hyperbolic type. As has already been mentioned, the determination of w within the circle reduces to the solution of a Laplace equation. This problem outside the Mach cone must be separately solved. Both solutions join at the Mach cone in such a way that on passing through it the velocity changes continuously.

Proceed to the transformation of equation (1.1). It is reduced to the equation of Laplace in three variables for the independent variables

$$x^* = ix\sqrt{M^2 - 1} \qquad y^* = iy\sqrt{M^2 - 1} \qquad z^* = z \tag{2.1}$$

In spherical coordinates r, σ, and ϑ where

$$x^* = r \cos \sigma \sin \vartheta \qquad y^* = r \sin \sigma \sin \vartheta \qquad z^* = r \cos \vartheta \tag{2.2}$$

equation (1.1) has the form

$$\frac{\partial}{\partial r}\left(r^2 F_r \sin \vartheta\right) + \frac{\partial}{\partial \sigma}\left(\frac{F_\sigma}{\sin \vartheta}\right) + \frac{\partial}{\partial \vartheta}\left(F_\vartheta \sin \vartheta\right) = 0$$

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Because of the constancy of the velocity along the rays issuing from 0, [1]

$$w_{\sigma\sigma} + \sin \vartheta \frac{\partial}{\partial \vartheta}\left(w_\vartheta \sin \vartheta\right) = 0 \qquad (2.3)$$

Within the Mach cone, imaginary transformations may be avoided by setting $\vartheta = i\theta$; then equations (2.1), (2.2), and (2.3) are replaced by

$$\xi = A \cos \sigma \operatorname{th} \theta \qquad \eta = A \sin \sigma \operatorname{th} \theta \qquad (2.4)$$

$$w_{\sigma\sigma} + \operatorname{sh} \theta \frac{\partial}{\partial \theta}\left(w_\theta \operatorname{sh} \theta\right) = 0 \qquad (2.5)$$

In order to reduce equation (2.5) to the Laplace equation in polar form, there remains to be introduced a new independent variable ϵ, determined from the condition $d\epsilon/d\theta = \epsilon/\operatorname{sh} \theta$; whence

$$\epsilon = \operatorname{th} \frac{\theta}{2} \qquad w_{\sigma\sigma} + \epsilon \frac{\partial}{\partial \epsilon}\left(w_\epsilon \epsilon\right) = 0 \qquad (2.6)$$

The same result may be repeated for u and v but not for φ because $\varphi_r \neq 0$. Thus w is the real part of a certain function of a complex variable, which is denoted by

$$Af\left(\epsilon e^{i\sigma}\right) = Af\left(\tau\right) = w + is$$

From equations (2.4) and (2.6), it follows that to transform the circle with center at 0 and radius A in the plane ξ,η (interior of the Mach cone) on the unit circle of the τ plane with center at the origin of coordinates, the radii vectors, keeping the polar angles σ unchanged, must be transformed by the formula

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[1] This condition may be expressed in the form

$$xw_x + yw_y + zw_z = 0 \quad \text{whence} \quad x^*w_{x^*} + y^*w_{*y} + z^*w_{z^*} = rw_z = 0$$

$$R = \sqrt{\xi^2 + \eta^2} = A \operatorname{th} \theta = \frac{2A \operatorname{th} \theta/2}{1 + \operatorname{th}^2 \theta/2} = \frac{2A\epsilon}{1 + \epsilon^2} \qquad (2.7)$$

3. COMPUTATION OF VELOCITY COMPONENTS ALONG X- AND Y-AXES

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Having the function $f(\tau)$, by integration the following equation may be obtained:

$$\omega = u + iv = -\frac{1}{2} \int \left(\tau \, df + \frac{\overline{df}}{\overline{\epsilon}} \right) \qquad (3.1)$$

This equation is obtained as a result of setting up $d\omega = \omega_\xi \, d\xi + \omega_\eta \, d\eta$ and transforming to the variables τ, $\overline{\tau}$. The partial derivatives ω_ξ and ω_η are computed with the aid of the relations $u_x = v_y$, $u_z = w_x$, $v_z = w_y$, and equation (1.1). It may be noted incidentally that according to the analogy of Busemann the magnitudes u, v, w correspond to x, y, $-\psi$ of Chaplygin and equation (3.1) agrees essentially with that of Chaplygin (reference 2, ch. V, eq. (94)).

4. BOUNDARY CONDITIONS

Consider the boundary conditions that are encountered in the problem of the arrow-shaped wing. If the wing does not extend beyond the Mach cone, then on the circle $|\tau| = 1$ the additional velocities u, v, w are equal to 0.[2] Assume a plane wing forming an infinitely small angle σ with the z-axis and an angle σ with the x-axis. The normal velocity $\operatorname{Im}(\omega e^{-i\sigma})$ along it will have a constant value. Whence using equation (3.1), the following is obtained:

$$\operatorname{Im}(\omega e^{-i\sigma}) = -\frac{1}{2} \operatorname{Im} \left\{ \frac{1}{A} \int \left[\frac{1}{\epsilon} (dw - ids) + \epsilon (dw + ids) \right] \right\} = \text{const}$$

$$ds = 0 \qquad (4.1)$$

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[2]Along those parts of the Mach cone where $dw = 0$, u and v will likewise be constant because from equation (3.1) for $\tau = e^{i\sigma}$ it follows that $d\omega = -\frac{1}{2} (e^{i\sigma} ids - ids/e^{-i\sigma}) = 0$.

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Let parts of the arrow-shaped plane wing inclined at an infinitely small angle to the x,z plane extend beyond the Mach cone (fig. 2). The region of influence of the extending parts of the wing fills in space the interiors of the Mach cones drawn from each point of the wing. In the ξ,η plane, this region is represented by ACDBEF, which is bounded by the tangents AC, AE, DB, and BF drawn from the edges of the wing to the Mach cone and the arcs CD and EF.

The flow outside the Mach cone evidently consists of plane flows. Hence, in each of the regions ACG, AGE, DBH, and FHB the velocities and the pressures will maintain constant values, whereas on passing through the wing w will change sign. For a wing symmetrical relative to the axis η, the boundary conditions for $f(\tau)$ on the corresponding arcs of the circle will be

$$\left.\begin{aligned} w &= w_0 \quad \text{on} \quad CG \quad \text{and} \quad HD \\ w &= -w_0 \quad \text{on} \quad GE \quad \text{and} \quad FH \\ w &= 0 \quad \text{on} \quad CD \quad \text{and} \quad EF \end{aligned}\right\} \tag{4.2}$$

5. ARROW-SHAPED WING EXTENDING BEYOND MACH CONE

Only a plane arrow-shaped wing with the arrow angle γ (fig. 1) symmetrical with respect to the plane $x = 0$, forming with the plane $y = 0$ a small angle β and finally cut along a straight line, which shall be assumed approximately coinciding with the axis $\eta = 0$, is considered.

The analysis begins with the case where the ends of the wing extend outside the Mach cone ($\tan\gamma > A$). The function $f(\tau)$ can easily be constructed from its singularities.

$$w + is = Af(\tau) = \frac{i\omega_0}{\pi} \lg \frac{\tau^2 - e^{-2i\sigma_0}}{\tau^2 e^{-2i\sigma_0} - 1} \tag{5.1}$$

It is not difficult to be convinced of the validity of equation (5.1) by direct check of conditions (4.1) and (4.2) because the analytic character of $Af(\tau)$ within the circle $|\tau| = 1$ is evident.

The concepts w_0 and σ_0 must be expressed in terms of the given magnitudes, the angle of attack β (or, what amounts to the same thing, in terms of the vertical velocity on the wing $v_0 = -W\beta$), and the cone angle γ. From figures 1 and 2 it is readily seen that

$$\tan \gamma = \frac{A}{\cos \sigma_0} \tag{5.2}$$

The velocity v_0 may be expressed in terms of w_0 by integration of equation (3.1):

$$v_0 = -\frac{1}{2} \operatorname{Im} \int_i^0 \left(\tau - \frac{1}{\tau}\right) \frac{df}{d\tau} d\tau = -\frac{w_0 \sin \sigma_0}{\pi A} \operatorname{Im} \left\{ \left[\lg \frac{(\tau - e^{-i\sigma_0})(\tau - e^{i\sigma_0})}{(\tau + e^{-i\sigma_0})(\tau + e^{i\sigma_0})} \right]_i^0 \right\}$$

whence

$$w_0 = -\frac{Av_0}{\sin \sigma_0} \quad \text{or} \quad \frac{w_0}{W} = \frac{\beta A}{\sin \sigma_0} \tag{5.3}$$

The lift-force coefficient of the wing is now computed.

$$C_y = \frac{2Y}{\rho W^2 S} = \frac{2}{W \tan \gamma} \int_{-\tan \gamma}^{\tan \gamma} w \, d\xi \tag{5.4}$$

where Y is the lift force, ρ the density of the gas, and $S = \tan \gamma$ is the area of the wing. Noting that along the parts of the wing extending outside the Mach angle cone, $w = \pm w_0$, the following is obtained:

$$C_y = \frac{2}{W \tan \gamma} \left[2w_0(\tan \gamma - A) + \int_{-1}^{1} w d\left(\frac{2A\tau}{1+\tau^2}\right) \right]$$

or integrating by parts

$$C_y = \frac{4w_0}{W \tan \gamma} \left[\tan \gamma - A \int_{-1}^{1} \frac{\tau}{1+\tau^2} \frac{d(w+is)}{w_0} \right] \tag{5.5}$$

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and finally

$$C_y = \frac{4w_0}{W \tan \gamma}\left[\tan \gamma - \frac{4A}{\pi} \sin 2\sigma_0 \int_{-1}^{1} \frac{\tau^2}{1+\tau^2} \frac{d\tau}{(\tau^2-e^{-2i\sigma_0})(\tau^2-e^{2i\sigma_0})}\right] \tag{5.6}$$

The integral entering equation (5.6) is most conveniently computed by deforming the contour of integration into an upper semicircle of unit radius passing around the poles $\tau_1 = e^{i\sigma_0}$, $\tau_2 = -e^{i\sigma_0}$, and $\tau_3 = i$ over infinitesimally small semicircles. From equation (5.5) and the boundary conditions on the Mach cone, it is easy to see that the integrals over the arcs of the upper semicircle $|\tau| = 1$ will be imaginary and evidently cancel each other. The computation of C_y thus reduces to the finding of the half-residues at the points τ_1, τ_2, and τ_3. By computation

$$\int_{-1}^{1} \frac{\tau}{1+\tau^2} \frac{d(w+ts)}{w_0} = \frac{1}{\cos \sigma_0} - \tan \sigma_0$$

whence making use of equations (5.6) and (5.3),

$$C_y = 4\beta A = \frac{4\beta}{\sqrt{M^2-1}} \tag{5.7}$$

This equation agrees with the well-known formula of Ackeret (reference 3) for the plane wing, which is a particular case of the arrow-shaped wing (for $\gamma = \pi/2$).

The distribution of the lift force, or more accurately of the magnitude $w/(AW\beta)$ proportional to the intensity of the lift force for the arrow-shaped wing extending outside the Mach cone, is shown in figure 3 (curve II). On the parts of the wing outside the Mach cone, the intensity of the lift force is constant. Within the Mach cone, curve II was computed by equation (5.8) obtained from equation (5.1):

$$\frac{w}{AW\beta} = \frac{1}{\sin \sigma_0}\left(1 - \frac{2}{\pi} \arctan \frac{\cos \sigma_0}{\sin \sigma_0} \frac{1-\tau^2}{1+\tau^2}\right) \qquad (5.8)$$

On the axis of abscissas the values of ξ referred to half the wing span were laid off. The Mach cones in figure 3 are represented in the form of the dotted semicircles. The nearer the Mach cone approaches the edges of the wing, the more intense is the lift force at the edges (for $\tan \gamma/A = 1$, curve III, at the edges of the wing $w/(AW\beta) = \infty$). For $A/\tan \gamma = 0$ (case of Ackeret), the intensity of the lift force along the wing is constant. Curve IV gives the distribution of the lift-force intensity for a wing within the Mach cone. (See section 6.) In this case at the edges of the wing, $w/(AW\beta) = \infty$.

6. ARROW-SHAPED WING LOCATED WITHIN MACH CONE

The case where the wing does not extend beyond the Mach cone ($\tan \gamma < A$) is now considered. In the plane τ, the wing is represented by the segment of the real axis from $-b$ to b where according to equation (2.7)

$$\tan \gamma = \frac{2Ab}{1+b^2} \qquad (6.1)$$

Conditions (4.1) and (4.2) in the plane τ assume the form: $w = 0$ on the circle, and $s = 0$ on the wing.

From the symmetry of the boundary conditions for $Af(\tau)$, it follows that in passing through the wing w changes sign. The function $Af(\tau)$ may be continued by the principle of mirror reflection over the entire plane with cuts from $-\infty$ to $-1/b$, from $-b$ to b, and from $1/b$ to ∞ (fig. 4). It is simplest to construct $w+is = Af(\tau)$ from the singular points and the zeros:

$$w+is = B \frac{\tau^2+1}{\sqrt{(b^2-\tau^2)(1/b^2-\tau^2)}} = Af \qquad (6.2)$$

where B denotes a real constant to be determined. Equation (6.2) can also be checked directly. The value of v_0 is found on the plate. For this value, the integration in equation (3.1) is carried out over the imaginary radius. Because along the axis of imaginaries $\tau = -\overline{\tau}$ and $-d\overline{f}/d\overline{\tau} = df/d\tau$; hence

$$(\omega)_{\tau=0} = -\frac{1}{2}\int_1^0 \frac{\tau^2-1}{\tau}\frac{df}{d\tau}\,d\tau = -B\,\frac{(b+1/b)^2}{2A}\int_0^1 \frac{(1-\tau^2)^2\,d\tau}{(b^2-\tau^2)^{3/2}(1/b^2-\tau^2)^{3/2}}$$

whence

$$v_0 = -\frac{B(b+1/b)^2}{2A}\int_0^1 \frac{(1+\epsilon^2)^2\,d\epsilon}{(b^2+\epsilon^2)(1/b^2+\epsilon^2)\sqrt{(b^2+\epsilon^2)(1/b^2+\epsilon^2)}} \tag{6.3}$$

setting

$$\epsilon = b\tan\varphi \qquad \sqrt{1-b^4} = k \qquad b^2 = k' \qquad \tan\varphi_1 = 1/\sqrt{k'} \tag{6.4}$$

the integral on the right side of equation (6.3) is readily transformed to the canonical form

$$v_0 = -\frac{Bb(b+1/b)^2}{2A}\int_0^{\varphi_1} \frac{\left[1-(1-k')\sin^2\varphi\right]^2\,d\varphi}{(1-k^2\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}$$

or

$$v_0 = -\frac{B}{2A\sqrt{k'}}\left[k'^2\int_0^{\varphi_1}\frac{d\varphi}{(1-k^2\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}} + 2k'F(\varphi_1,k)+E(\varphi_1,k)\right]$$

where $F(\varphi_1,k)$ and $E(\varphi_1,k)$ are elliptic integrals of the first and second kind. As is known (ref. 4, equation (126))

$$\int_0^{\varphi_1}\frac{d\varphi}{(1-k^2\sin^2\varphi)^{3/2}} = \frac{E(\varphi_1,k)}{k'^2} - \frac{k^2}{k'^2}\,\frac{\sin\varphi_1\cos\varphi_1}{\sqrt{1-k^2\sin^2\varphi_1}} = \frac{E(\varphi_1,k)}{k'^2} - \frac{1-k'}{k'^2}$$

whence

$$-\frac{B}{Av_0} = \frac{\sqrt{k'}}{k'F(\varphi_1,k) + E(\varphi_1,k) - (1-k')/2} \tag{6.5}$$

With the aid of equation (6.5) from equation (6.2), and from equation (5.1) with the aid of equation (5.3), it is found that in the limiting case when the wing touches the Mach cone ($\sigma_o = 0$, $b = 1$)

$$w+is = \frac{2v_0 A}{\pi} \frac{\tau^2+1}{\tau^2-1} \tag{6.2a}$$

The lift-force coefficient of the wing is now computed. From equation (5.4) transforming with the aid of equation (2.7) to the variable ϵ,

$$C_y = \frac{4A}{W \tan \gamma} \int_{-b}^{b} w \frac{1-\epsilon^2}{(1+\epsilon^2)^2} \, d\epsilon \tag{6.6}$$

The integral entering equation (6.6) may be replaced by a contour integral about the wing.

$$C_y = \frac{2A}{W \tan \gamma} \int (w+is) \frac{1-\tau^2}{(1+\tau^2)^2} \, d\tau \tag{6.7}$$

This integral is most simply computed by deforming the contour of integration into a unit circle passing around the poles at the points $\tau = \pm i$ over infinitesimally small semicircles. Because on the circle w is equal to 0, it is easily seen that the integrals over the arcs of the circle will be imaginary and will mutually cancel. The computation of C_y thus reduces to finding the semiresidues at the points $\tau = \pm i$.

$$C_y = \frac{2AB}{W \tan \gamma} \int \frac{1-\tau^2}{\sqrt{(b^2-\tau^2)(1/b^2-\tau^2)}} \frac{d\tau}{1+\tau^2} = \frac{2\pi B}{W}$$

whence

$$\frac{C_y}{\beta A} = \frac{2\pi \sqrt{k'}}{k'F(\varphi_1,k) + E(\varphi_1,k) - (1-k')/2} \tag{6.8}$$

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From equations (5.7), (6.1), and (6.8), it is seen that $C_y\sqrt{M^2 - 1}/(4\beta)$ is a function of $\sqrt{M^2 - 1}\tan\gamma$. The computations (fig. 5) can be carried out by assigning various values to the parameter k. The center of pressure of the wing is at the center of gravity of the triangle.

7. REMARKS ON DRAG

If the wing extends beyond the Mach cone, the velocities at the edges of the wing will be finite and the efficiency of the wing $Y/X = 1/\beta$, where X is the drag. In the case where the wing is entirely within the Mach cone, then as a result of the infinite velocities at the leading sharp edges suction forces may be expected, which increase the efficiency of the wing.

The drag of the wing can be computed with the aid of the momentum theorem applied to the volume of gas enclosed in the cylinder about the wing and the Mach cone, one of the bases of the cylinder lying in the ξ,η plane and the other to the left of the origin of coordinates in the region of the undisturbed flow.

$$X = \frac{\rho}{2}\iint\left[u^2 + v^2 + (M^2-1)\,w^2)\right]\,d\xi\,d\eta \qquad (7.1)$$

where the integration is extended over the entire base of the cylinder in the ξ,η plane. In deriving equation (7.1), use is also made of the Bernoulli integral, in which after expanding in powers of u, v, w all terms were neglected that contained the additional velocities to degrees higher than the second, the adiabatic condition, and the equation of continuity in integral form.

Translated by S. Reiss,
National Advisory Committee
for Aeronautics

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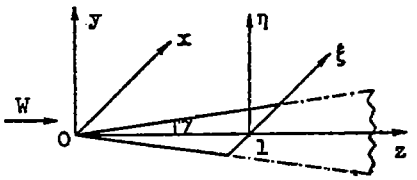

Figure 1.

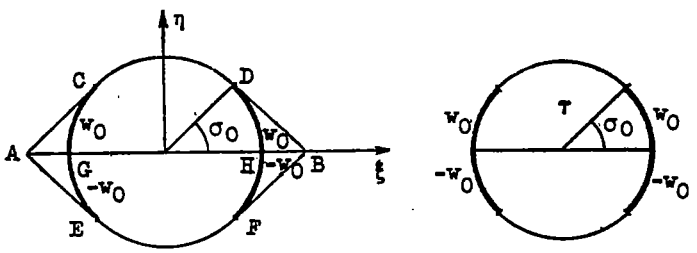

Figure 2.

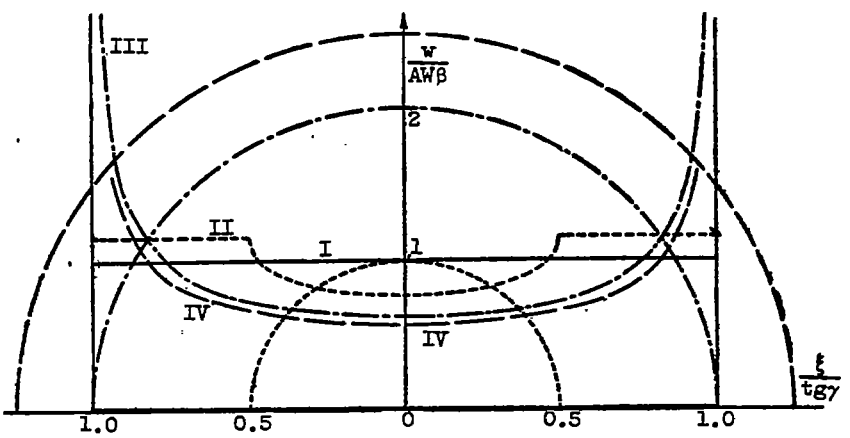

Figure 3.

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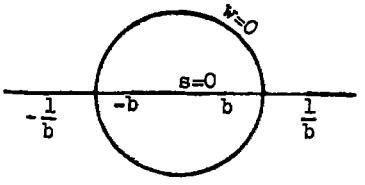

Figure 4.

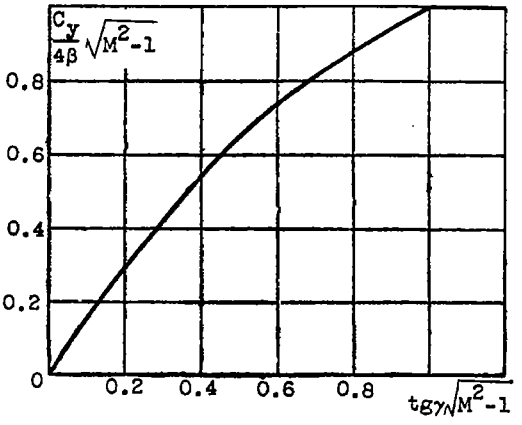

Figure 5.